Integrated Nonlinear Structural Analysis and Design

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An integrated approach to the design optimization of structures that requires nonlinear analysis is proposed. The optimization process begins with a linearized structural response, and the amount of nonlinearity is increased as one gets closer to the optimum design. The proposed approach has the potential of reducing substantially the computational cost of the optimization. Two truss examples are used for demonstration.

Introduction

MOST of the work in structural optimization has been limited to constraints based on linear analysis of the structure. In the past few years, there has been interest in the design of structures in the nonlinear response regime (e.g., Refs. 1–3). Because a single nonlinear analysis of a complex structure can be very expensive computationally, the cost of the optimization process can become prohibitive.

Nonlinear structural analysis is iterative in nature, with the solution typically obtained for a succession of increasing loads up to the design load. Because the optimization process is also iterative, it may be possible to integrate the two iterative processes and improve efficiency. One approach to such integration was proposed in Ref. 4, where a simultaneous analysis and design procedure was proposed. The response variables were treated as design variables along with the structural parameters, and the equations of equilibrium were treated as equality constraints.

The approach of Ref. 4 is attractive for poorly banded three-dimensional problems where the use of Gaussian elimination to solve the iterative nonlinear analysis is not practical. For problems where Gaussian elimination is effective, it may be desirable to retain it and seek other approaches for integrating the analysis and design iterations.

Wu and Arora[5] proposed a design procedure that employs sensitivity analysis to obtain a first approximation to the nonlinear response of a modified structure. Newton's method is then used to converge to the exact response. This approach substantially reduces the need to retrace repeatedly the nonlinear load displacement trajectory as the structure is being designed.

The present paper continues in the same direction of employing sensitivity information to obtain a first approximation to the nonlinear response of a modified structure. This is accompanied by the idea of introducing the nonlinearity of the problem gradually through the design process. At the initial stages of the design the structure is analyzed at low load levels where its behavior is approximately linear, and then its response is scaled up to the design load. This procedure permits complete integration of the analysis and design iterations.

The proposed procedure is limited to problems where the response is stable (no bifurcations) and path-independent. Two truss problems are used to demonstrate its computational potential. However, much more computational experience is needed to assess its effectiveness for more general problems.

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Sequential Optimization Procedure

The optimization of a structure subject to stress and displacement constraints can be written as

$$\text{minimize } f(X)$$
$$\text{such that } g_j(U,X) \geq 0 \quad j = 1,\ldots,n_g \tag{1}$$

where X is a vector of design variables, U a displacement vector, f an objective function, and g_j constraint functions. The displacement field is calculated from nonlinear analysis, which typically solves the equations of equilibrium,

$$R(U,X) = \lambda P(X) \tag{2}$$

in terms of a load parameter λ. The solution proceeds by incrementing λ from zero to the design value λ_d in n_l load steps. One of the most popular methods for solving Eq. (2) is the modified Newton method. Given $U(\lambda_i)$ at the ith load step, $U(\lambda_{i+1})$ is found by the iterative procedure

$$K_T\left[U^{(k+1)} - U^{(k)}\right] = -\lambda_{i+1}P(X) + R(U^{(k)},X) \tag{3}$$

where $U^{(k)}$ is the kth iterate approximation to $U(\lambda_{i+1})$, K_T (the tangential stiffness matrix) is the Jacobian of R, $\partial R/\partial U$ evaluated at $U(\lambda_i)$, and $U^{(o)} = U(\lambda_i)$. The derivatives of U with respect to the components of X are obtained by differentiating Eq. (2) with respect to x_i:

$$K_T\frac{\partial U}{\partial x_i} = \lambda\frac{\partial P}{\partial x_i} - \frac{\partial R}{\partial x_i} \tag{4}$$

where the Jacobian K_T is evaluated at the current load level.

Sequential Approximate Optimization

Because of the high cost of the nonlinear analysis, a common procedure is to use the derivatives of the response to construct approximations to the constraints. Denoting approximate quantities by subscript a, we have

$$U_a(\lambda,X) = U(\lambda,X_o) + \Delta U\left(X,X_o,\frac{\partial U}{\partial X}\right) \tag{5}$$

In the present work, a linear approximation is used so that

$$\Delta U = \sum_{i=1}^{n}\frac{\partial U}{\partial x_i}(x_i - x_{oi}) \tag{6}$$

However, other first-order approximations (e.g., Refs. 6 and 7) are also popular.

Then constraints are approximated as

$$g_{aj}(U,X) = g_j(U_a,X) \tag{7}$$

The optimization problem is then replaced by

$$\begin{aligned} &\text{minimize } f(X) \\ &\text{such that } g_{aj}(U,X) \geq 0 \quad j = 1,\ldots,n_g \\ &\text{and } \mid x_i - x_{oi} \mid \leq e_i \quad i = 1,\ldots n \end{aligned} \tag{8}$$

where e_i is chosen so as to guarantee the accuracy of the approximation U_a. The optimum obtained from the solution of Eq. (8) is then used as the next X_o for the approximation, and the process is repeated to convergence. Often the move limits e_i have to be gradually shrunk to prevent oscillations once we reach the neighborhood of the optimum design. A flow chart of the process is shown in Fig. 1. This commonly used optimization procedure is employed as a standard against which the efficiency of the proposed integrated approach is measured.

Integrated Analysis and Design

The proposed integration of the analysis and design processes is based on a secant extrapolation of $U(\lambda_d,X)$ from the solution at a lower load level (see Fig. 2):

$$U(\lambda_d,X) \cong \frac{\lambda_d}{\lambda_i} U(\lambda_i,X) \tag{9}$$

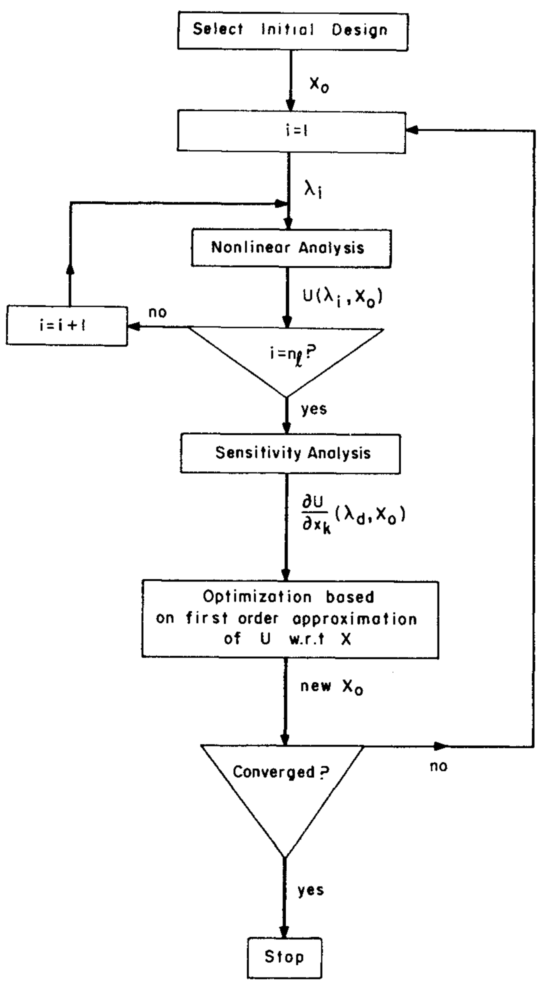


Fig. 1 Flow chart for sequential approximate optimization.

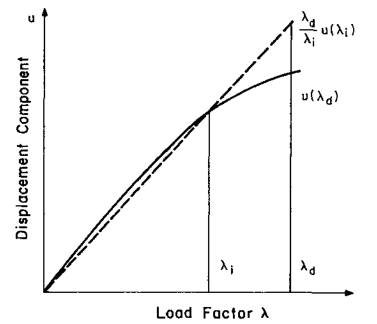


Fig. 2 Secant approximation of displacement component at design load factor λ_d based on value at load factor λ_i.

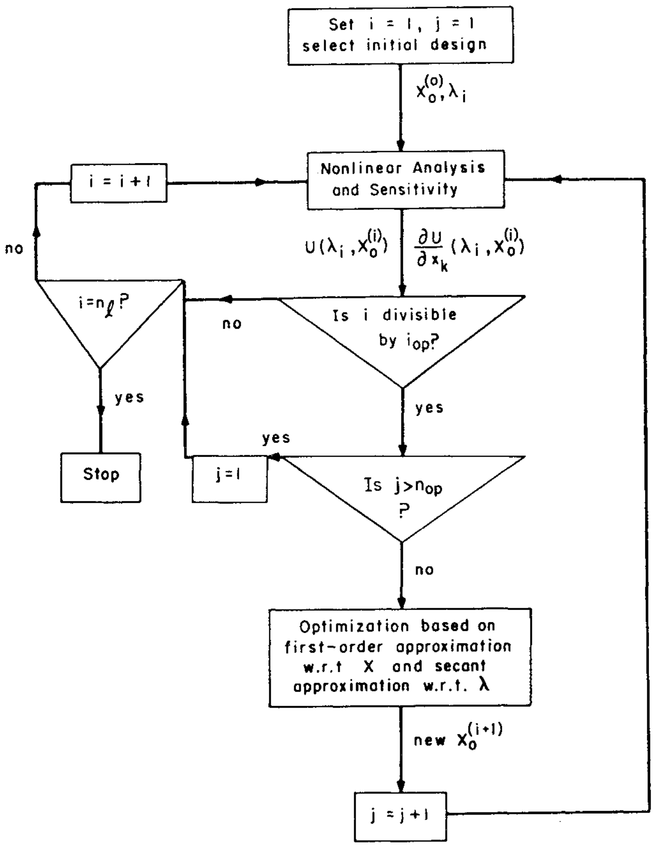
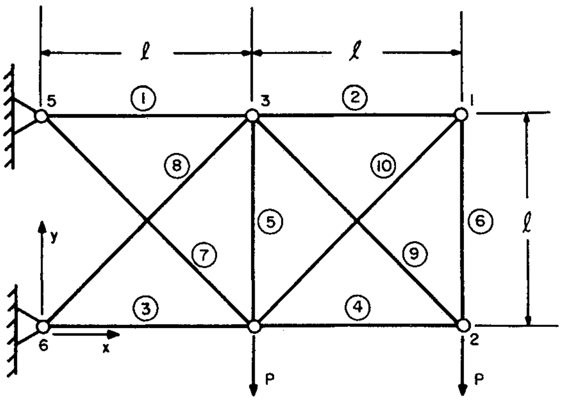


Fig. 3 Flow chart for integrated nonlinear analysis and approximate optimization (n_ℓ = number of load steps, i_{op} = number of load steps per optimization, n_{op} = number of optimizations per load step).

By using Eq. (9), the structure can be optimized based on the analysis at any load level, and we integrate the optimization and load iterations as shown in Fig. 3. As shown in Fig. 3, the load iteration becomes the outer loop in the process. At each load step we may wish to perform one or more approximate optimization iterations. We start the design with an initial design $X_0^{(o)}$ and calculate the response at the first load level $U(\lambda_1,X_0^{(o)})$. If optimization is indicated at that load level, we also calculate derivatives of the response with respect to the design variables. Then we perform one or more approximate optimization iterations, leading to a new design, which is denoted as $X_0^{(1)}$. (After each approximate optimization, the stiffness matrix for the new design is regenerated and factored and the displacement field and its derivatives are recomputed.) The

Table 1 Initial and final designs for 10-bar truss example, E = 100 ksi (689.5 MPa)

	Initial design		Final Design	
Member	Area, in.^{2} (cm^2)	Stress, ksi (MPa)	Area, in.^{2} (cm^2)	Stress, ksi (MPa)
1	8 (51.61)	22.29 (153.7)	5.744 (37.06)	25.00 (172.4)
2	8 (51.61)	3.58 (24.7)	0.500 (3.23)	5.70 (39.3)
3	8 (51.61)	−14.10 (−97.2)	0.500 (3.23)	−19.86 (−136.9)
4	8 (51.61)	−1.29 (−8.9)	0.500 (3.23)	4.02 (27.7)
5	8 (51.61)	5.52 (38.1)	0.979 (6.32)	25.00 (172.4)
6	8 (51.61)	3.80 (26.2)	0.500 (3.23)	6.55 (45.1)
7	8 (51.61)	18.56 (128.0)	3.689 (23.80)	25.00 (172.4)
8	8 (51.61)	−17.43 (−120.2)	5.376 (34.68)	−25.00 (−172.4)
9	8 (51.61)	10.14 (69.9)	3.833 (24.73)	25.00 (172.4)
10	8 (51.61)	−4.87 (−33.6)	0.500 (3.23)	−6.81 (−46.9)



ℓ = 360", P = 100 kips

Fig. 4 Ten-bar truss.

Table 2 Convergence of design for 10-bar truss using traditional sequential approximate optimization

Design iteration	Volume, in.^{3} (m^3)	Maximum stress, ksi (MPa)
0	33571 (0.5501)	22.29 (153.7)
1	23974 (0.3929)	25.06 (172.8)
2	18824 (0.3084)	25.20 (173.8)
3	15205 (0.2492)	25.70 (177.2)
4	13345 (0.2187)	25.00 (172.4)
5	11922 (0.1954)	24.99 (172.3)
6	10868 (0.1781)	25.50 (175.8)
7	10290 (0.1686)	25.01 (172.4)
8	9964 (0.1633)	25.41 (175.2)
9	9935 (0.1628)	25.24 (174.0)
10	9962 (0.1632)	25.00 (172.4)

Total for 10 design iterations:
Number of optimizations, $n_{oT} = 10$
Number of stiffness matrix factorizations, $n_F = 400$
Number of Newton iterations, $n_I = 1095$

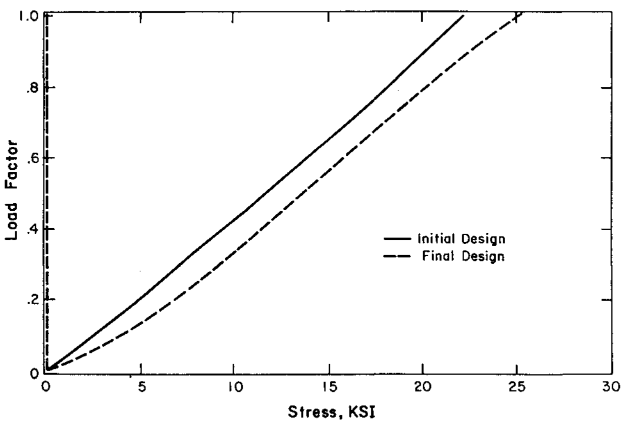


Fig. 5 Ten-bar truss, stress in element 1 for initial and final design.

Table 3 Computational performance of integrated analysis and design procedure for various control parameters, 10-bar truss

n_ℓ	i_{op}	n_{op}	n_{oT}	n_F	n_I	Final volume, in.^{3} (m^3)
21	3	3	21	35	123	9963 (0.1633)
18	3	3	18	30	111	9966 (0.1633)
15	3	3	15	25	101	10162 (0.1665)
24	3	2	16	32	115	10121 (0.1659)
21	3	2	14	28	100	10242 (0.1678)
20	2	2	20	30	110	9964 (0.1633)
18	2	2	18	27	105	9965 (0.1633)
16	2	2	16	24	97	10122 (0.1659)
32	2	1	16	32	108	10099 (0.1655)
30	2	1	15	30	103	10122 (0.1659)
10	1	2	20	20	96	9964 (0.1633)
9	1	2	18	18	88	9965 (0.1633)
8	1	2	16	16	86	10120 (0.1658)

Legend:
n_ℓ = number of load steps
i_{op} = number of load steps per optimization
n_{op} = number of optimizations per load step
n_F = number of stiffness matrix factorizations
n_I = number of Newton iterations
n_{oT} = number of approximate optimizations

superscript denotes the load level, and the zero subscript indicates that this design will be used as the basis of approximations, using Eq. (5).

Next, the load level is incremented and $U(\lambda_2, X_0^{(1)})$ is calculated. The zeroth iterate (first guess) to $U(\lambda_2, X_0^{(1)})$ is taken to be $U(\lambda_1, X_0^{(o)})$. As indicated in Ref. 5, it is probably better to use as zeroth iterate the approximation to $U(\lambda_1, X_0^{(1)})$ given by Eq. (5). However, for the examples tested in this work the use of this approximation did not result in any improvement in convergence.

The integrated procedure is characterized by three numbers, n_ℓ, i_{op}, and n_{op}. As shown in Fig. 3, the structure is optimized at every i_{op} load step (e.g., if $i_{op} = 2$ at every other load step) n_{op} times. Thus, the total number of approximate optimizations is $(n_\ell / i_{op}) \times n_{op}$. The goal of the integrated approach is to choose the given parameters so as to reduce the cost of the nonlinear analyses required for the optimization to the cost of a single analysis.

Examples

The two truss examples analyzed were selected because they are often used as demonstration problems in structural optimization problems. Only geometrical nonlinearity was considered, and to enhance the nonlinearity, Young's modulus was reduced to the point where about eight loading steps were required for completing the analysis (i.e., with fewer load

Table 4 Computational performance of integrated analysis and design procedure with n_{Lop} optimizations performed for last load step (with one optimization per load step up to last step), 10-bar truss

n_ℓ	n_{Lop}	n_{oT}	n_F	n_I	Final volume, in.^{3} (m^3)
9	4	12	12	64	9967 (0.1633)
9	3	11	11	63	10061 (0.1649)
8	5	12	12	63	9967 (0.1633)
8	4	11	11	62	9973 (0.1634)
8	3	10	10	60	10087 (0.1653)
7	5	11	11	62	9965 (0.1633)
7	4	10	10	61	10001 (0.1639)
7	3	9	9	60	10127 (0.1659)
6	5	10	10	62	9965 (0.1633)
6	4	9	9	61	10037 (0.1645)
6	3	8	8	59	10180 (0.1668)
5	5	9	9	68	9982 (0.1636)
5	4	8	8	66	10102 (0.1655)
4	6	9	9	77	9965 (0.1633)
4	5	8	8	75	10013 (0.1641)

Legend:
n_ℓ = number of load steps
n_{oT} = total number of optimizations
N_F = number of stiffness matrix factorizations
n_I = number of Newton iterations

Table 6 Convergence of design for 72-bar truss using traditional sequential approximate optimization

Design iteration	Volume, in.^{3} (m^3)	Maximum stress, ksi (MPa)
0	1706.2 (0.02795)	150.59 (1038.3)
1	1653.0 (0.02709)	70.67 (487.3)
2	1732.3 (0.02839)	53.46 (368.6)
3	1868.7 (0.03062)	32.24 (222.3)
4	1958.5 (0.03209)	26.76 (184.5)
5	1950.8 (0.03197)	26.35 (181.7)
6	1964.6 (0.03219)	25.05 (172.7)
7	1963.9 (0.03218)	25.05 (172.7)
8	1964.5 (0.03219)	25.00 (172.4)

Total for eight design iterations:
Number of optimizations, $n_{oT} = 8$
Number of stiffness matrix factorization, $n_F = 64$
Number of Newton iterations, $n_I = 312$
Total CPU time (IBM 3084) = 24.4 s

Table 5 Initial and final design for 72-bar truss, $E = 400$ ksi (2758 MPa)

Design variable group no.	Members	Initial design: Cross-sectional area, in.^{2} (cm^2)	Initial design: Maximum stress in group, ksi (MPa)	Final design: Cross-sectional area, in.^{2} (cm^2)	Final design: Maximum stress in group, ksi (MPa)
1	1-4	0.2 (1.290)	83.83 (578.0)	0.356 (2.297)	25.00 (172.4)
2	5-12	0.2 (1.290)	20.38 (140.5)	0.223 (1.439)	−25.00 (−172.4)
3	13-16	0.2 (1.290)	−18.29 (−126.1)	0.163 (1.052)	−25.00 (−172.4)
4	17-18	0.2 (1.290)	−32.70 (−225.5)	0.294 (1.897)	−25.00 (−172.4)
5	19-22	0.2 (1.290)	96.10 (662.6)	0.576 (3.716)	25.00 (172.4)
6	23-30	0.2 (1.290)	−26.11 (−180.0)	0.185 (1.194)	25.00 (172.4)
7	31-34	0.2 (1.290)	−6.43 (−44.3)	0.100 (0.645)	5.24 (36.1)
8	35-36	0.2 (1.290)	−8.40 (−57.9)	0.100 (0.645)	9.61 (66.3)
9	37-40	0.2 (1.290)	120.27 (829.3)	0.882 (5.690)	25.00 (172.4)
10	41-48	0.2 (1.290)	−24.10 (−166.2)	0.178 (1.148)	25.00 (172.4)
11	49-52	0.2 (1.290)	−9.22 (−63.6)	0.100 (0.645)	5.21 (35.9)
12	53-54	0.2 (1.290)	−8.14 (−56.1)	0.100 (0.645)	3.45 (23.8)
13	55-58	0.2 (1.290)	150.59 (1038.3)	0.117 (0.755)	25.00 (172.4)
14	59-66	0.2 (1.290)	−36.67 (−252.8)	0.181 (1.168)	25.00 (172.4)
15	67-70	0.2 (1.290)	15.47 (106.7)	0.100 (0.645)	2.96 (20.4)
16	71-72	0.2 (1.290)	2.33 (16.1)	0.100 (0.645)	2.92 (20.1)

steps, the modified Newton's method diverged). The convergence criterion for the analysis was that the right-hand side of Eq. (3) was reduced in norm to 0.001 times the norm of P.

Ten-Bar Truss

The 10-bar truss example, Fig. 4, was optimized subject to a maximum stress limit of 25,000 psi (172 MPa) and a minimum gage constraint of 0.5 in.^{2} (3.23 cm^2). The initial design for all cases was 8 in.^{2} (51.61 cm^2) for all members.

Figure 5 shows the dependence of the stress in member 1 on the load for the initial and final designs (given in Table 1). It is clear that the behavior of the final design is more nonlinear than the initial design. Curiously, the nonlinearity is more pronounced at lower loads. Because of the high nonlinearity of the final design, 40 load steps were required for the analysis of that design while 10 were sufficient for the initial design. The total number of Newton iterations, n_I, was about 60 for a wide range of number of load steps.

Table 2 shows the convergence of the design using the traditional sequential approach. Move limits of ±30% were used for each optimization and an IMSL routine (ZX4LP) was used to solve the linear approximate optimization problem. Because of the slow convergence for the final design 40 load steps were used throughout, which accounts for the large number of stiffness matrix factorizations ($n_F = 400$) and Newton iterations ($n_I = 1095$). It is estimated that if the number of load steps would have been changed with the design, n_F and n_I could have been reduced to about $n_F = 10$ and $n_I = 600$. Because of the small size of the analysis problem, the computational cost was dominated by the optimization (about 1 CPU s on an IBM 3084 per optimization).

Next, the integrated approach was exercised for various combinations of the number of load steps n_ℓ, number of load steps per optimizations i_{op}, and number of optimizations per load step n_{op}. The results are summarized in Table 3. They show that the number of stiffness matrix factorizations n_F can be reduced to 20-30 and the total number of Newton iterations n_I to about 100. These numbers are three to six times smaller than the estimated minimum for the sequential approach. However, the number of approximate optimizations required to get within one or two percent of the optimum volume increased to 15-20.

The increase in the number of optimizations was diagnosed to be due to the need for fine-tuning the design at the last load step. Consequently, another approach was taken where only one optimization was performed at intermediate load steps and N_{Lop} optimizations were performed at the last load steps. A more robust approach would be to iterate for the last load step until the design is converged. The results are summarized in Table 4.

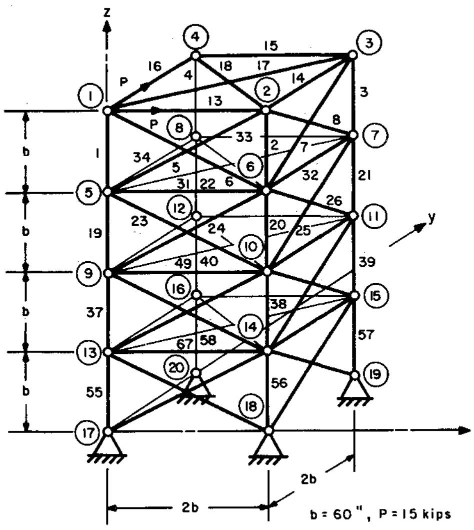


Fig. 6 Seventy-two-bar truss.

Table 7 Convergence of design for 72-bar truss using partial sequential approximate optimization (one approximation per one unconstrained minimization)

Design iteration	Volume, in.^{3} (m^3)	Maximum stress, ksi (MPa)
0	1706.2 (0.02796)	150.59 (1038.3)
1	1948.1 (0.03192)	67.67 (466.6)
2	2161.5 (0.03542)	35.62 (245.6)
3	2053.6 (0.03365)	28.16 (194.2)
4	1965.4 (0.03221)	−30.83 (−212.6)
5	1972.5 (0.03232)	25.73 (177.4)
6	1970.2 (0.03228)	−25.13 (−173.2)
7	1964.6 (0.03219)	25.10 (173.1)
8	1964.4 (0.03219)	25.00 (172.4)

Total for eight design iterations:
Number of optimizations, $n_{oT}=1$
Number of stiffness matrix factorizations, $n_F=64$
Number of Newton iterations, $n_I=285$
Total CPU time (IBM 3084) = 14.22 s

It is clear from Table 4 that by performing several optimizations only for the last load steps the total number of optimizations n_{oT} can be reduced to about 10, the number of matrix factorizations n_F to 10, and the number of Newton iterations n_I to 60–70. As noted at the beginning of this section, these numbers are typical of the cost of a single analysis. Thus, we achieved the goal of reducing the computational effort associated with the analysis to that of a single nonlinear analysis.

Seventy-Two-Bar Truss

The 72-bar truss, Fig. 6, was optimized subject to a maximum stress limit of 25,000 psi (172 MPa) and a minimum gage constraint of 0.1 in^2 (0.254 cm^2). This time the initial design was chosen to be overstressed, with all members having an area of 0.2 in^2 (0.508 cm^2). Sixteen design variables, corresponding to the standard linking for that structure, were employed. The initial and final designs are defined in Table. 5.

When the approximate optimization was performed with the ZX4LP routine, the optimization time was about 6 s of IBM 3084 CPU time compared with about 1.5 s for the analysis and derivative calculation. The excessive time for the optimization may be due to ZX4LP not being programmed to deal efficiently with problems with large numbers of constraints. The optimization was then performed with the NEWSUMT-A program,[8] which required only about 1.5 CPU s per optimization. This compared with about 1.3 s for the nonlinear analysis (eight load steps, $n_F=8$, $n_I=39$) and 0.15 s for calculating derivatives of stresses with respect to design variables. The history of the design convergence (with move limits of 30%) is shown in Table 6.

NEWSUMT-A is based on a penalty function algorithm that requires repeated unconstrained minimizations for successively smaller values of the penalty multiplier. The program allows the use of approximations inside the optimization iteration, with one exact analysis performed for each penalty multiplier. This approach was taken next for the traditional sequential optimization, and the results are summarized in Table 7. The total cost was reduced to 14.2 CPU s, with the optimization accounting for 2.5 s, derivative calculation for 1.2 s, and eight analyses requiring 10.5 s. Next, the integrated approach was implemented. Based on the results from the 10-bar truss, one optimization was performed for each load step, except for the final one. Results with various combinations of numbers of load steps and optimizations are summarized in Table 8. As in Table 7, each optimization is a partial one corresponding to a constant value of the penalty multiplier. It is seen that the optimum design can be obtained with about 7–9 matrix factorizations and about 60–65 Newton iterations. This is about the same number of factorizations and 50% more in Newton iterations than a *single* nonlinear analysis. For a typical case, $n_\ell=6$, $n_{Lop}=3$, the optimization required 1.9 s, derivative calculations 1.2 s, and the analysis 1.3 s.

Concluding Remarks

An integrated analysis and optimization approach for the design of structures subject to constraints on nonlinear static response was presented. The procedure was demonstrated on two truss problems subject to stress and minimum gauge constraints. It was found that the analysis cost required for the design could be reduced to close to that of a single nonlinear

Table 8 Computational performance of integrated analysis and design procedure with n_{Lop} partial optimizations performed for last load step, 72-bar truss

n_ℓ	n_{Lop}	n_F	n_I	CPU time, s	Final volume, in.^{3} (m^3)	Maximum stress, ksi (MPa)
8	2	9	62	5.00	1962.3 (0.03216)	25.07 (172.9)
8	1	8	59	4.48	1961.5 (0.03214)	25.19 (173.7)
7	3	9	62	5.07	1963.3 (0.03217)	25.02 (172.5)
7	2	8	60	4.54	1962.3 (0.03216)	25.08 (172.9)
7	1	7	57	4.21	1962.0 (0.03215)	25.20 (173.8)
6	4	9	63	4.93	1963.5 (0.03218)	25.01 (172.4)
6	3	8	61	4.43	1963.6 (0.03218)	25.02 (172.5)
6	2	7	59	4.16	1963.1 (0.03217)	25.05 (172.7)
6	1	6	56	3.70	1963.6 (0.03218)	25.20 (173.8)
5	4	8	68	4.44	1963.4 (0.03217)	25.01 (172.4)
5	3	7	66	3.97	1963.5 (0.03218)	25.03 (172.6)
5	2	6	64	3.51	1962.1 (0.03215)	25.11 (173.1)

Legend:
n_ℓ = number of load steps
n_F = number of stiffness matrix factorizations
n_I = number of Newton iterations

analysis. While these preliminary results are encouraging, extensive testing with more complex problems is required before general conclusions can be drawn.

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